MATHEMATICS

Maximum Marks: 80
Time Allotted: Three Hours
Reading Time: Additional Fifteen Minutes

Instructions to Candidates

1. You are allowed an **additional fifteen minutes** for **only** reading the paper.
2. You must **NOT** start writing during reading time.
3. The question paper has **14 printed pages**.
4. It consists of **twenty questions** and **three sections: Sections A, B, C and D**.
5. All questions are **compulsory.**
6. **Section A** consists of **very short answer questions** of **1 mark** each.
7. **Section B** consists of **short answer questions** of **2 marks each**.
8. **Section C** consists of **moderately long answer questions** of **3 marks each**.
9. **Section D** consists of **long answer questions** of **5 marks each**.
10. Internal choices have been provided in **three** questions of 2 marks each, **three** questions of 3 marks each and **three** questions of 5 marks each.
11. While attempting **Multiple Choice Questions in Section A**, you are required to **write only ONE option as the answer.**
12. The intended marks for questions or parts of questions are given in the brackets [].
13. All workings, including rough work, should be done on the same page as, and adjacent to, the rest of the answer.
14. Mathematical tables and graph papers are provided.

Instruction to Supervising Examiner

Kindly read aloud the instructions given above to all the candidates present in the examination hall.

Note: The Specimen Question Paper in the subject provides a realistic format of the Board Examination Question Paper and should be used as a practice tool. The questions for the Board Examination can be set from any part of the syllabus, though the format of the Board Examination Question Paper will remain the same as that of the Specimen Question Paper.

SECTION A – 20 MARKS

Question 1

In subparts (i) to (xvii) choose the correct options and in subparts (xviii) to (xx), answer the questions as instructed.

(i) A relation R is defined on $\mathbb{Z}$ as aRb if and only if $a^2 - 7ab + 6b^2 = 0$. Then R is: **[1]** **(Understand)**

(a) reflexive and symmetric

(b) transitive but not reflexive

(c) symmetric but not reflexive

(d) reflexive but not symmetric

(ii) If $\theta = sin^{-1}x + cos^{-1}x - tan^{-1}x,\ x \geq 0$, then the smallest interval in which θ lies is: **[1]** **(Analyse)**

(a) $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$

(b) $0 < \theta < \pi$

(c) $\frac{-\pi}{4} \leq \theta \leq 0$

(d) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

(iii) Let A be the area of a triangle having vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3). Which of the following is correct? **[1]** **(Understand)**

(a) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = +A$

(b) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2A$

(c) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm \frac{A}{2}$

(d) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = A^2$

(iv) A particle moves along the curve $6y = x^3 + 2$. At what point(s) on the curve, is the y-coordinate changing 8 times as fast as the x-coordinate? **(Apply)** **[1]**

(a) $\left(2, \frac{5}{3}\right)$ only

(b) (4,11) only

(c) (4,11) and $\left(-4, \frac{-31}{3}\right)$

(d) (8,86) and (−8, −85)

(v) **Statement I**: If a satellite's altitude function $h(t)$ has a positive derivative $(h'(t) > 0)$, the satellite is moving away from the Earth. **[1]**

Statement II: At the apogee (highest point), the instantaneous velocity ($\frac{dh}{dt}$) of the satellite relative to its altitude is zero.

Which of the following is correct? **(Understand)**

(a) Statement I is true and Statement II is false.

(b) Statements I is false and Statement II is true.

(c) Both the statements are true.

(d) Both the statements are false.

(vi) For what value(s) of k do the tangents of two curves $x = y^2$ and $xy = k$ cut at right angles? **(Understand)** **[1]**

(a) $k = \pm\frac{1}{4}$

(b) $k = \pm 1$

(c) $k = 0$

(d) $k = \pm\frac{1}{2\sqrt{2}}$

(vii) Evaluate the nature of the point (0,0) for the curve $y = x^4$, given that $f''(0) = 0$. **(Understand)** **[1]**

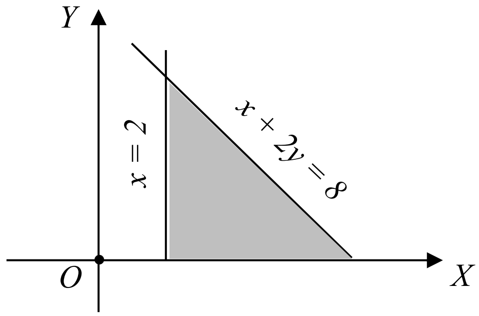
(a) The function at the point (0,0) is undefined because the second derivative test fails.

(b) It is a point of local maximum because the first derivative is changing its sign from positive to negative as x increases through 0.

(c) It is a local minimum because the second derivative is positive for all $x \neq 0$.

(d) It is a point of neither local maximum nor local minimum because the second derivative is zero.

(viii) The expression for finding the area of the shaded region is: **[1]**



(Apply)

(a) $\int_2^8 \left(4 - \frac{x}{2} - 2\right) dx$

(b) $\int_2^8 \left(\frac{x}{2} + 4\right) dx$

(c) $\int_2^8 \left(\frac{y}{2} + 4\right) dy$

(d) $\int_2^8 \left(4 - \frac{x}{2}\right) dx$

(ix) Shown below is a solved anti-differentiation problem to obtain $f(x)$: **[1]**

$\frac{d}{dx} f(x) = \frac{1}{x(\log x)^2}$ such that $f(e) = -1$

Taking anti-derivative:

$f(x) = \int \frac{1}{x(\log x)^2} dx + C$

Step 1 $\Rightarrow f(x) = \int \frac{d(\log x)}{(\log x)^2} + C$

Step 2 $\Rightarrow f(x) = -\frac{1}{\log x} + C$

Given $f(e) = -1$

$\Rightarrow f(e) = -1 + C$

$\Rightarrow C = 0$

Step 3 $f(x) = -\frac{1}{\log x}$

In which step is there an error (if any) in the solution? **(Understand)**

(a) Step 1

(b) Step 2

(c) Step 3

(d) No error

(x) On solving $\int -\frac{3}{\sqrt{1-x^2}}dx$, Anil's answer was $\int -\frac{3}{\sqrt{1-x^2}}dx = 3\cos^{-1}x + c$ and Jaspreet's answer was $\int -\frac{3}{\sqrt{1-x^2}}dx = -3\int \frac{dx}{\sqrt{1-x^2}} = -3\sin^{-1}x + c$. **[1]**

Whose answer was correct? **(Recall)**

(a) Only Anil's answer was correct.

(b) Only Jaspreet's answer was correct.

(c) Both of them were correct.

(d) Neither of them was correct.

(xi) If $(\vec{a}) = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$, $(\vec{b}) = -\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$, $(\vec{c}) = \hat{\imath} + \hat{\jmath} + \hat{k}$, then $(\vec{a} \times \vec{b}).(\vec{a} \times \vec{c})$ is: **[1]**

(Recall)

(a) 60

(b) 64

(c) 74

(d) -74

(xii) The point of intersection of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{3-z}{-4}$ and $\frac{x-1}{5} = \frac{2-y}{-2} = \frac{z-3}{1}$ is: **[1]**

(Understand)

(a) $(1, 2, -3)$

(b) $(-1, 2, 3)$

(c) $(1, -2, 3)$

(d) $(1, 2, 3)$

(xiii) An objective function $Z = ax + by$ is maximum at points (8,10) and (7,12). If $a, b \geq 0$ and ab = 72, then the maximum value of the function is equal to: **[1]**

(Apply)

(a) 84

(b) 132

(c) 156

(d) 176

(xiv) **Statement I:** Every Linear Programming Problem has at least one optimal solution. **[1]**

Statement II: If a Linear Programming Problem has two optimal solutions, then it has infinitely many solutions.

Which of the statements is correct? **(Understand)**

(a) Statement I is true and Statement II is false.

(b) Statements I is false and Statement II is true.

(c) Both the statements are true.

(d) Both the statements are false.

(xv) If $P(A) = m, P\left(\frac{B}{\bar{A}}\right) = 3m, P\left(\frac{B}{A}\right) = 6m,$ then what will be $P\left(\frac{A}{B}\right)$? **(Analyse)** **[1]**

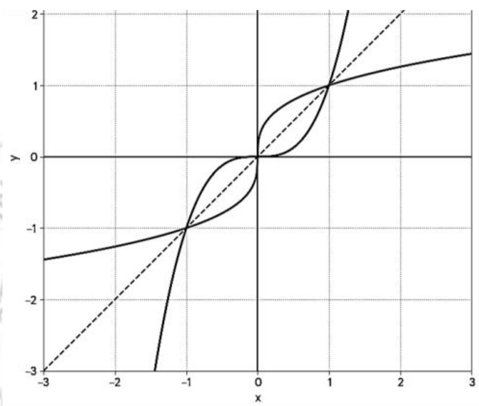
(a) $\dfrac{2m}{1+m}$

(b) $\dfrac{4m}{1+m}$

(c) $\dfrac{7m}{1+m}$

(d) $\dfrac{9m}{1+m}$

(xvi) Observe the given graph: **[1]**



Assertion: The curves given above can be considered as $f(x)$ and $f^{-1}(x)$.

Reason: The curves are reflections of each other across the line $x = y$.

Which one of the following is correct? **(Understand)**

(a) Both Assertion and Reason are true, and Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.

(c) Assertion is true and Reason is false.

(d) Assertion is false and Reason is true.

(xvii) **Assertion:** $\int_{-1}^{1} x^3 \cos x \, dx = 0$ **[1]**

Reason: If $f(x)$ is a continuous function defined on $[0, a]$, then

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Which one of the following is correct? **(Analyse)**

(a) Both Assertion and Reason are true, and Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.

(c) Assertion is true and Reason is false.

(d) Assertion is false and Reason is true.

(xviii) Using the properties of matrices, explain why the following concept is incorrect: **[1]**

Since $(a+b)(a-b) = a^2 - b^2$, therefore, (A + B) (A – B) = $A^2 - B^2$ also must be true if A and B are matrices. **(Understand)**

(xix) If the solution of the differential equation $\frac{dy}{dx} = \frac{ax+3}{2y+5}$ represents a circle, then find the value of a. **(Analyse)** **[1]**

(xx) The probability distribution of random variable X is given below. **[1]**

X	1	2	3	4	5
P(X)	m	3m	a	5m	b

If $P(X \leq 2) = 0 \cdot 28$ and $P(X \geq 4) = 0 \cdot 52$, find $P(X = 3)$. **(Apply)**

SECTION B – 14 MARKS

Question 2 **[2]**

A straight line L_θ has vector equation $\vec{r} = 5\hat{\imath} + \lambda\left(5\hat{\imath} + \sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k}\right)$ and a plane π_P has equation $x = p$, $p \epsilon \mathbb{R}$.

Show that the angle between L_θ and π_P is independent of both θ and p. **(Understand)**

Question 3 **[2]**

In a school, for a period of 8 working days, it is equally likely that Tanishka is present or absent on any given day.

What is the probability that she is present in school for at least 5 consecutive working days? **(Evaluate)**

Question 4 **[2]**

Solve the following differential equation:

$$\frac{dx}{dy} = \frac{x}{y} + \frac{f\left(\frac{x}{y}\right)}{f'\left(\frac{x}{y}\right)}$$ **(Apply)**

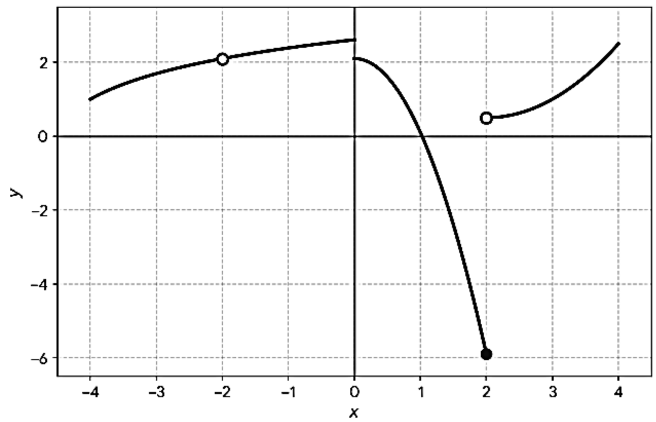
Question 5

(i) Determine the values of constants p and q such that the function **[2]**

$f(x) = \begin{cases} p\sin x + q, \forall x \leq 0 \\ x^2 + 2x + 1, \forall x > 0 \end{cases}$ is differentiable at $x = 0$. **(Understand)**

OR

(ii) Observe the graph given below:



(a) State the value(s) of x where the function has removable discontinuity. **[1]**

(Understand)

(b) State the value(s) of x where the graph of the function is not differentiable. **[1]**

(Understand)

Question 6 **[2]**

(i) Parag is working on a school project on right-angled triangles. He draws a right-angled triangle PQR with ∠R = 90°. The sides opposite angles P, Q, R are p, q, r respectively.

Evaluate the expression: $tan^{-1}\left(\frac{p}{q+r}\right) + tan^{-1}\left(\frac{q}{r+p}\right)$. **(Apply)**

OR

(ii) The equation given below has equal roots:

$ax^2 + sin^{-1}(x^2 - 2x + 2) + cos^{-1}(x^2 - 2x + 2) = 0$

What is the value of 'a'? **(Apply)**

Question 7 **[2]**

(i) Find the vector projection of $\vec{B} = 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ on $\vec{A} = \hat{\imath} - 2\hat{\jmath} - 2\hat{k}$ and the scalar component of $\vec{B}$ on $\vec{A}$. **(Apply)**

OR

(ii) If the position vectors of three points A, B, C are respectively $\hat{\imath} + \hat{\jmath} + \hat{k}$, $2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ and $7\hat{\imath} + 4\hat{\jmath} + 9\hat{k}$, find the unit vector perpendicular to the plane of triangle ABC. **(Apply)**

Question 8

Three landmarks of Dehradun are joined by straight roads. *Clock tower* of Dehradun is considered as the 'origin'. *IMA (I)* is 3 km east and 9 km north of *Clock Tower* and *Rajpur (R)* is 5 km east and 5 km south of *IMA*. A *bus stop (S)* is situated two thirds of the way along the road from Clock Tower to IMA.

Considering $\hat{\imath}$ as a 1 km vector pointing east and $\hat{\jmath}$ as a 1 km vector pointing north:

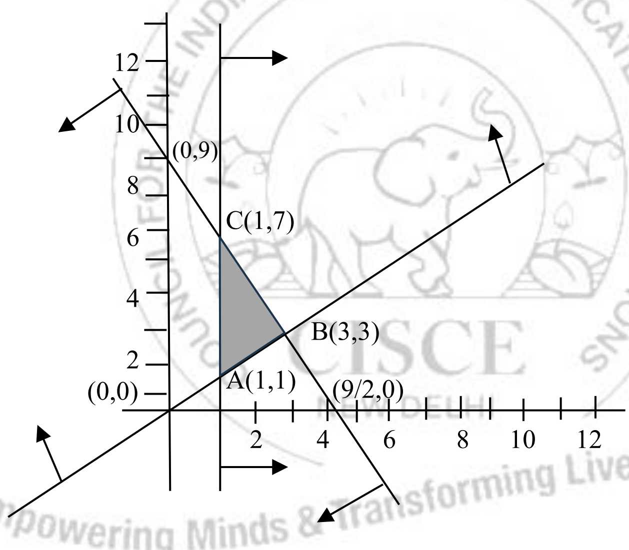
(i) Find the position vector of the *bus stop (S)* relative to the *Clock Tower*. **(Understand)** **[1]**

(ii) Prove that the *bus stop (S)* is the closest point to *Rajpur (R)* on the *Clock Tower* to *IMA(I)* Road. **(Understand)** **[1]**

SECTION C – 21 MARKS

Question 9

The feasible region determined by some constraints is represented by the shaded region in the graph given below:



(i) Formulate the constraints which represent the above feasible region. **(Understand)** **[1]**

(ii) Hence, maximise the objective function given by $Z = x + y$. **(Understand)** **[1]**

(iii) What change in the constraints will make the feasible region unbounded? **(Understand)** **[1]**

Question 10 **[3]**

If $x = 3z + 1$ and $y = f(x)$, then prove that $9\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}$ **(Apply)**

Question 11

(i) Find the derivative of the function $f(x) = \log\left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right] + \tan^{-1}\frac{2x}{1-x^2}$ with respect to x. **(Evaluate)** **[2]**

(ii) An analyst claims that the function $f(x)$ has a local maximum at some point $x = c$. Evaluate this claim using the expression for $f'(x)$. **(Evaluate)** **[1]**

Question 12

(i) A vector $\vec{n}$ of magnitude 8 units is inclined to x - axis at 45°, y - axis at 60° and at an acute angle with z-axis. A plane through the point $\left(\sqrt{2}, -1, 1\right)$ is normal to $\vec{n}$. Find the equation of the plane in vector form. **(Analyse)** **[3]**

OR

(ii) The planes π_1 and π_2 have equations $2x + 6y - 2z = 5$ and $3x + 9y + pz = -\frac{51}{2}$ respectively.

(a) Verify that the point P $\left(2, \frac{1}{2}, 1\right)$ lies on the plane π_1. **(Apply)** **[1]**

(b) Determine the value of p if π_2 is parallel to π_1. **(Analyse)** **[1]**

(c) A line through P normal to π_1 meets π_2 at the point Q. Find the coordinates of Q. **(Analyse)** **[1]**

Question 13 **[3]**

(i) If x, y, z are distinct non zero real numbers, prove that

$$\Delta = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} y^2 & z^2 & x^2 \\ y^3 & z^3 & x^3 \\ 1 & 1 & 1 \end{vmatrix}.$$

Hence or otherwise evaluate Δ in its simplest form if $xy + yz + zx = 1$. **(Analyse)**

OR

(ii) A school is organising its Annual Day function and plans to decorate every chair with a ribbon and every table with a cover. There are 150 chairs and 20 tables, all of which need decorations. Company A charges ₹ 12 per chair ribbon and ₹ 180 per table cover. Company B charges ₹ 10 per chair ribbon and ₹ 200 per table cover.

By setting up one matrix equation that include both companies, compare the overall prices that would be charged by the two companies. **(Analyse)**

Question 14

Let $\int_1^5 3f(x)dx = 12$

(i) Show that $\int_5^1 f(x)dx = -4$. **(Understand)** **[1]**

(ii) Find the value of $\int_1^2 (x + f(x))\, dx + \int_2^5 (x + f(x))dx$. **(Understand)** **[2]**

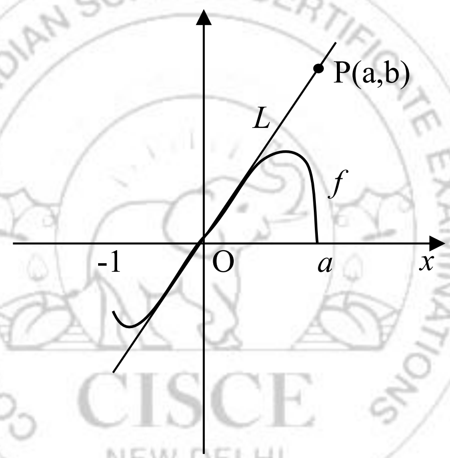
Question 15

(i) The diagram below shows the graph of $f(x) = 2x\sqrt{a^2 - x^2}$, for $-1 \le x \le a$, where $a > 1$. **[3]**

The line L is the tangent to the graph of $f(x)$ at the origin O.

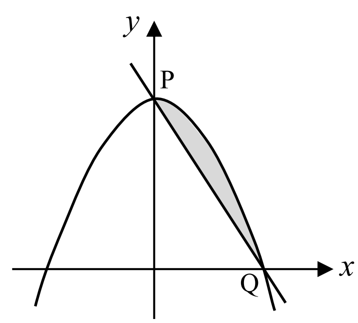
Given that $f'(x) = \frac{2a^2 - 4x^2}{\sqrt{a^2 - x^2}}$, for $-1 \le x \le a$.

Using integration, find the area of ΔOPQ in terms of a. **(Apply)**



OR

(ii) A line with equation $y = -3x + 9$ intersects the axes at the points P and Q. A parabola of the form $y = ax^2 + c$, where $a , c \in \mathbb{Z}$, also passes through the points P and Q as shown in the diagram below.



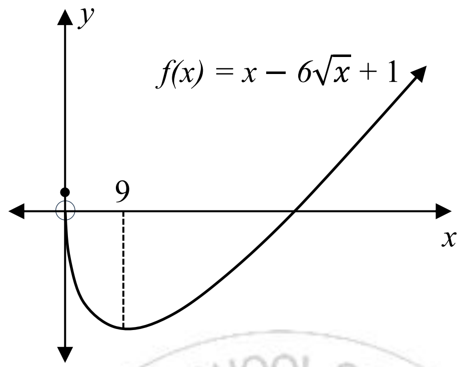
(a) Obtain the equation of the parabola. **(Apply)** **[1]**

(b) Using integration find the area of the shaded region. **(Apply)** **[2]**

SECTION D – 25 MARKS

Question 16

The graph of $f(x) = x - 6\sqrt{x} + 1$ is shown below.



(i) State the natural domain of $f(x)$. **(Analyse)** **[1]**

(ii) Does $f(x)$ have an inverse function? Explain your answer. **(Analyse)** **[1]**

(iii) Let $g(x) = x - 6\sqrt{x} + 1, 0 \leq x \leq 9$. Find $g^{-1}(x)$ and verify $(g \circ g^{-1})(4) = 4$. **(Analyse)** **[1]**

(iv) Let $h(x) = x - 6\sqrt{x} + 1, x \geq 9$. Find $h^{-1}(x)$ and verify $(h \circ h^{-1})(16) = 16$. **(Analyse)** **[1]**

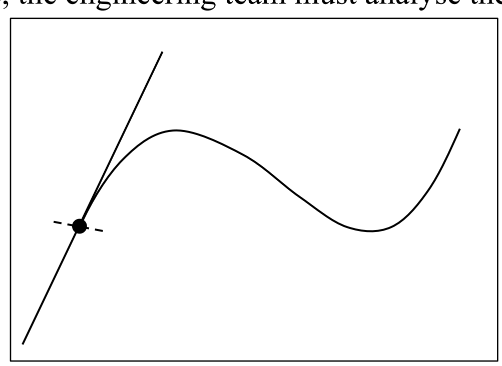
(v) Find the value of x such that $g^{-1}(x) = h^{-1}(x)$ **(Analyse)** **[1]**

Question 17

(i) An engineering team is designing a section of a new roller coaster track. The vertical profile of the track for a specific horizontal stretch is modelled by the function:

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$$

where x represents the horizontal distance from the start of the section (in meters) and $f(x)$ represents the height of the track (in meters). To ensure safety and a thrilling experience, the engineering team must analyse the steepness and the peaks of this track.



(a) Identify the intervals of x where the roller coaster is climbing (increasing height) and where it is descending (decreasing height). **(Analyse)** **[1]**

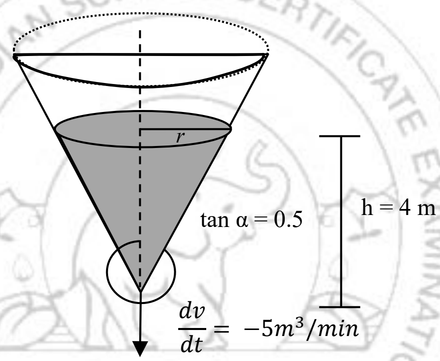
(b) A support beam must be placed at the point where the track's slope is exactly zero. Find the coordinates of the points where support beam must be placed. **(Evaluate)** **[1]**

(c) Determine the maximum and minimum heights reached by the roller coaster in the interval $x \in [0, 4]$. **(Evaluate)** **[2]**

(d) At the point $x = 1$, a maintenance ladder must be placed perpendicular to the track, (along the normal). Find the equation of the maintenance ladder at $x = 1$. **(Evaluate)** **[1]**

OR

(ii) A large industrial water tank is shaped like an inverted right circular cone with a semi-vertical angle of $\tan^{-1}(0.5)$. Water is being drained out for a manufacturing process at a constant rate of $5\ \text{m}^3/\text{min}$.



(a) Find the rate at which the water level is dropping when the height of the water is 4 meters. **(Evaluate)** **[2]**

(b) Show that the wetted surface area of the tank $S(h)$ is a strictly increasing function of the water depth h. **(Analyse)** **[1]**

(c) A chemical additive must be added to the interior surface of the water tank. If the cost of the additive is proportional to the square of the surface area $(C = KS^2)$, find the depth h at which the cost is increasing most rapidly relative to time. **(Evaluate)** **[2]**

Question 18

(i) (a) Evaluate: $\int \ln x\ dx$. **(Understand)** **[2]**

(b) Hence, evaluate: $\int \frac{\ln\left[\ln\left(\frac{1+x}{1-x}\right)\right]}{1-x^2} dx$ **(Evaluate)** **[3]**

OR

(ii) (a) Evaluate: $\int \tan x\ dx$ **(Understand)** **[2]**

(b) Hence, evaluate: $\int \frac{dx}{\cot\frac{x}{2}\cot\frac{x}{3}\cot\frac{x}{6}}$ **(Evaluate)** **[3]**

Question 19

(i) In a game show, a contestant is shown three closed doors: Door A, Door B and Door C. Behind one door is a laptop while the other two doors have nothing behind them. The laptop is placed randomly, so each door is equally likely to contain the laptop.

The contestant first selects Door A. The host, who knows where the laptop is and never opens the door containing the laptop, opens Door B and reveals that there is nothing behind the door.

The host then asks whether the contestant would like to choose to Door A or change to Door C.

(a) Find the probability that the laptop is behind Door A, given that Door B has been opened. **(Analyse)** **[2]**

(b) Find the probability that the laptop is behind Door C, given that Door B has been opened. **(Analyse)** **[2]**

(c) Hence, state giving reasons, whether the contestant should change their selection or not. **(Analyse)** **[1]**

OR

(ii) In a large organisation, only 1 out of every 1,500 emails contains a harmful attachment. An automated filtering system is used to detect such emails.

The filtering system correctly marks a harmful email as *dangerous* 97% of the time. It also correctly marks a safe email as *safe* 97% of the time.

(a) Find the probability that the email is safe. **(Analyse)** **[1]**

(b) Find the probability that given the email is harmful the filter marks it *safe*. **(Analyse)** **[2]**

(c) One particular email has been flagged as *dangerous* by the system. Find the probability that this email is actually safe. **(Analyse)** **[2]**

Question 20 **[5]**

A ball is thrown vertically downwards from the top of a cliff, and its position is tracked from when it is first thrown until it hits the ground (at which point it may be assumed) that the ball comes instantaneously to rest.

The height of the ball above the ground after t seconds is given by the equation $S(t) = at^2 + bt + c$, where a, b and c are real constants and the height S is measured in meters. It is observed that after 1 second, the ball is 285 m above the ground; after 2 seconds, its height is 260 m and after 4 seconds it is 180 m.

Set up and solve a matrix equation to find the height of the cliff. **(Evaluate)**

MATHEMATICS

ANSWER KEY

SECTION A – 20 MARKS

Question 1

In answering Multiple Choice Questions, candidates have to write either the correct option number or the explanation against it. Please note that only ONE correct answer should be written.

(i) **(d) reflexive but not symmetric** **[1]**

Reflexive: $\forall\, a \in \mathbb{Z}$, we have,

$a^2 - 7a \cdot a + 6a^2 = a^2 - 7a^2 + 6a^2 = 0 \Rightarrow (a, a) \in R$

$\therefore$ Relation is reflexive

Symmetric:

For $(6,1)$ as, $6^2 - 7 \times 6 \times 1 + 6 \times 1^2 = 36 - 42 + 6 = 0 \Rightarrow (6,1) \in R$.

But, for (1,6): $1^2 - 7.1.6 + 6.6^2 = 1 - 42 + 216 \neq 0 \Rightarrow (1,6) \notin R$

$\therefore$ Relation is not symmetric.

(ii) **(d) $\boldsymbol{\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}}$** **[1]**

$sin^{-1}x$ and $cos^{-1}x$ is defined for $-1 \leq x \leq 1$

$tan^{-1}x$ is defined for all real x

Given, $x \geq 0$, the valid domain is: $0 \leq x \leq 1$

For all $x \in [-1, 1], sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

$\theta = \frac{\pi}{2} - tan^{-1}x$

The principal value range of $tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

But since $x \geq 0$, $tan^{-1}x \in 0 \left[0, \frac{\pi}{4}\right]$ $(\because x \leq 1)$

$0 \leq tan^{-1}x \leq \frac{\pi}{4}$

$\Rightarrow -\frac{\pi}{4} \leq -tan^{-1}x \leq 0$

$\Rightarrow \frac{\pi}{4} \leq \frac{\pi}{2} - tan^{-1}x \leq \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

(iii) **(b)** $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2$ **A** **[1]**

Area of triangle formed by three points is:

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since determinant may be positive or negative

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2 \text{ A}$$

(iv) **(c) (4,11) and** $\mathbf{(-4, -\frac{31}{3})}$ **[1]**

Differentiating w.r.t 't', $6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$.

Substituting $\frac{dy}{dt} = 8\ \frac{dx}{dt}$ gives the value of $x = \pm 4$.

When $x = 4, y = 11$ *and when* $x = -4, y = \frac{-31}{3}$

$\therefore$ Points are (4,11) and $(-4, -\frac{31}{3})$

(v) **(c) Both the statements are true.** **[1]**

Statement I: If $h'(t) > 0$, it means altitude increases with time, so satellite is moving away from earth

$\therefore$ Statement I is true.

Statement II: At highest point (apogee), altitude becomes maximum. At maximum point, $\frac{dh}{dt} = 0$.

This means change in altitude is zero.

$\therefore$ Statement II is true

(vi) **(d)** $\boldsymbol{k = \pm\frac{1}{2\sqrt{2}}}$ **[1]**

$x = y^2\ \Rightarrow 1 = 2y\frac{dy}{dx}\ \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

$xy = k\ \Rightarrow x\frac{dy}{dx} + y = 0\ \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

Since tangents of the given two curves are at right angles, product of their slopes $= -1$

$\Rightarrow \frac{1}{2y}.\frac{-y}{x} = -1\ \Rightarrow x = \frac{1}{2} = y^2$

$\therefore k^2 = x^2y^2\ \Rightarrow \frac{1}{8}$

$\therefore k = \pm\frac{1}{2\sqrt{2}}$

(vii) **(c) It is a local minimum because the second derivative is positive for all $\boldsymbol{x \neq 0}$.** **[1]**

$\frac{dy}{dx} = 4x^3 \Rightarrow \frac{d^2y}{dx^2} = 12x^2$

Though $\frac{d^2y}{dx^2} = 0$ when $x = 0$, the fact that $\frac{d^2y}{dx^2} = 12x^2$ is always positive for all $x \neq 0$ which confirms the curve is a concave up at $x = 0$.

The function attains local minimum at (0,0).

(viii) **(d) $\boldsymbol{\int_2^8 \left(4 - \frac{x}{2}\right) dx}$** **[1]**

Given equation of line: $x + 2y = 8$

$$\Rightarrow \frac{x}{8} + \frac{y}{4} = 1$$

$$\Rightarrow y = 4 - \frac{x}{2}$$

Required area is the area bounded by $x = 2;\ y = 4 - \frac{x}{2}$ and x-axis.

$\therefore$ Required area $= \int_2^8 \left(4 - \frac{x}{2}\right) dx$

(ix) **(d) No error** **[1]**

(x) **(c) Both of them were correct.** **[1]**

Notice that: $\cos\left(-\sin^{-1} x + \frac{\pi}{2}\right)$

$= \sin(\sin^{-1} x)$

$= x$

So, the answers are equivalent.

(xi) **(d) $\boldsymbol{-74}$** **[1]**

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = -10\hat{\imath} + 9\hat{\jmath} + 7\hat{k}$

$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4\hat{\imath} - 3\hat{\jmath} - \hat{k}$

$\therefore\ (\vec{a} \times \vec{b}).(\vec{a} \times \vec{c}) = -40 - 27 - 7 = -74$

(xii) **(d) $\boldsymbol{(1, 2, 3)}$** **[1]**

Given lines are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-1}{5} = \frac{y-2}{2} = \frac{z-3}{1}$

$\because$ Both lines are passing through (1,2,3)

$\therefore$ The point of intersection is (1,2,3).

(xiii) **(c) 156** **[1]**

Maximum at points (8,10) and (7,12)

$\Longrightarrow 8a + 10b = 7a + 12b \Longrightarrow a = 2b$

But as $ab = 72 \Longrightarrow 2b^2 = 72 \Longrightarrow b^2 = 36 \Longrightarrow b = 6.$

$\therefore a = 2b = 12$

Maximum value $= 8 \times 12 + 10 \times 6 = 156$

(xiv) **(b) Statement I is false and Statement II is true.** **[1]**

Statement I: Every Linear Programming Problem has at least one optimal Solution.

False - Some LPPs are infeasible

Statement II: If an Linear Programming Problem has two optimal solutions, then it has infinitely many solutions.

True - If two points give same maximum value, then all points on the line joining them also give same value.

Hence, infinitely many optimal solutions exist.

(xv) **(a)** $\boldsymbol{\frac{2m}{1+m}}$ **[1]**

$P\left(\frac{B}{A}\right) = 6m \Longrightarrow P(A \cap B) = P(A) \times 6m = 6m^2$

$P\left(\frac{B}{\bar{A}}\right) = 3m \Longrightarrow P(B \cap \bar{A}) = P(\bar{A}) \times 3m = 3m(1-m) = 3m - 3m^2$

$P(B) - P(A \cap B) = 3m - 3m^2 \Longrightarrow P(B) = 3m - 3m^2 + 6m^2 = 3m + 3m^2$

$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{6m^2}{3m+3m^2} = \frac{2m}{1+m}$

(xvi) **(a) Both Assertion and Reason are true, and Reason is the correct explanation of Assertion.** **[1]**

The assertion states that the curves can be considered as $f(x)$ and $f^{-1}(x)$ Since the reflection property is satisfied

∴ Assertion is true.

The reason states that the curves are reflections of each other across the line $x = y$.

The reflection of a curve across the line $y = x$ yields the graph of the inverse function because the geometric reflection swaps the x and y coordinates, which corresponds to the algebraic definition of an inverse mapping.

∴ Reason is true and Reason correctly explains the Assertion.

(xvii) **(b) Both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.** **[1]**

Let $f(x) = x^3 \cos x$

$f(-x) = -x^3 \cos x = -f(x)$

$\Rightarrow f(x)$ is an odd function

$\Rightarrow \int_{-1}^{1} x^3 \cos x \, dx = 0$

As $\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is an odd function.

Hence assertion is true.

If $f(x)$ is a continuous function defined on $[0, a]$, then

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$ is also true but not applicable for this assertion.

(xviii) Matrix Multiplication is non – commutative (AB ≠ BA) **[1]**

(A + B) (A – B) = $A^2 - AB + BA - B^2$

But AB ≠ BA

∴ (A + B) (A – B) ≠ $A^2 - B^2$

Hence, the concept is incorrect.

(xix) $\boldsymbol{a = -2}$ **[1]**

$\frac{dy}{dx} = \frac{ax+3}{2y+5}$

$\Rightarrow (2y+5)dy = (ax+3)dx$

Integrating both sides:

$\Rightarrow \int (2y+5)dy = \int (ax+3)dx$

$\Rightarrow y^2 + 5y + c = a\frac{x^2}{2} + 3x$

$\Rightarrow -a\frac{x^2}{2} - 3x + y^2 + 5y + c = 0$

Now if $a = -2$, then the solution of the differential equation will be

$\Rightarrow x^2 + y^2 - 3x + 5y + c = 0$, which is an equation of circle.

$\therefore a = -2$

(xx) **0·2** **[1]**

$P(X \leq 2) = 0 \cdot 28 \Longrightarrow 4m = 0 \cdot 28 \Longrightarrow m = 0 \cdot 07$

$P(X \geq 4) = 0 \cdot 52 \Longrightarrow 5m + b = 0 \cdot 52 \Longrightarrow b = 0 \cdot 52 - 0 \cdot 35 = 0 \cdot 17$

$9m + a + b = 1 \Longrightarrow a = 1 - 0 \cdot 63 - 0 \cdot 17 = 0 \cdot 2$

$P(X = 3) = 0 \cdot 2$

SECTION B – 14 MARKS

Question 2 **[2]**

$\because\ \vec{r} = 5\hat{\imath} + \lambda\left(5\hat{\imath} + \sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k}\right)$,

$\therefore$ Direction vector of above line $\vec{d} = 5\hat{\imath} + \sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k}$.

Now equation of plane π_P is $x = p$.

$\Rightarrow \vec{r}.\vec{n} = p$ where n is the normal vector to the plane.

$\therefore \vec{n} = \hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

$\Rightarrow \vec{n} = \hat{\imath}$

$\therefore \sin\theta = \frac{|\vec{d}.\vec{n}|}{|\vec{d}|.|\vec{n}|} = \frac{|5|}{\sqrt{26}.1}$

$\theta = \sin^{-1}\left(\frac{5}{\sqrt{26}}\right)$

Hence proved.

Question 3 **[2]**

On any given day, Tanishka can be either present or absent.

Total possible outcomes = 2^8

If we consider that she comes to school as **P** and she is absent as **A**,
Case 1: If Tanishka comes for **at least** the first 5 consecutive working days, then we just need to consider the attendance for the remaining 3 days.

PPPPP _ _ _

This can be done in 2^3 ways.

Case 2: If Tanishka comes for **at least** any other 5 consecutive working days, then she must be absent on the preceding day. So, we just need to consider the attendance for the remaining 2 days.

APPPPP_ _ or **_ APPPPP _** or **_ _ APPPPP**

This can be done in 2^2 ways.

$\therefore$ Probability (comes to school for at least 5 consecutive working days)

$= \frac{2^3+2^2+2^2+2^2}{2^8} = \frac{5}{64}$

Question 4 **[2]**

$\frac{dx}{dy} = \frac{x}{y} + \frac{f\left(\frac{x}{y}\right)}{f'\left(\frac{x}{y}\right)}$

Given differential equation is homogenous differential equation of type

$\frac{dx}{dy} = f\left(\frac{x}{y}\right)$

Substituting, $x = vy$ i.e. $\frac{x}{y} = v$

$\Rightarrow \frac{dx}{dy} = v + y.\frac{dv}{dy}$

$\Rightarrow v + y.\frac{dv}{dy} = v + \frac{f(v)}{f'(v)}$

$\Rightarrow y.\frac{dv}{dy} = \frac{f(v)}{f'(v)}$

$\Rightarrow \frac{f'(v)}{f(v)} dv = \frac{dy}{y}$

Integrating both sides

$\Rightarrow \int \frac{d(f(v))}{f(v)} = \int \frac{dy}{y} + \log c$

$\Rightarrow \log|f(v)| = \log y + \log c$

$\Rightarrow f(v) = cy$

$\Rightarrow f\left(\frac{x}{y}\right) = cy,$ which is the required solution.

Question 5

(i) Since differentiability implies continuity, the limits from both sides at $x = 0$ must be equal. **[2]**

$LHL = \lim_{x \to 0^-} p \sin x + q = q$

$RHL = \lim_{x \to 0^+} x^2 + 2x + 1 = 1$

At $x = 0, f(0) = q = LHL = RHL \implies q = 1$

The Left-Hand Derivative (LHD) must equal the Right-Hand Derivative (RHD) at $x = 0$.

$LHD = \frac{d}{dx}(p \sin x + q) = p \cos x,$

At $x = 0,\ LHD = p \cos x = p$

$RHD = \frac{d}{dx}(x^2 + 2x + 1) = 2x + 2$

At $x = 0, RHD = 2x + 2 = 2$

$\Rightarrow p = 2$

OR

(ii) (a) At $x = -2$ the function has removable discontinuity. **[1]**

At $x = -2$, LHL = RHL but $f(-2)$ is missing.

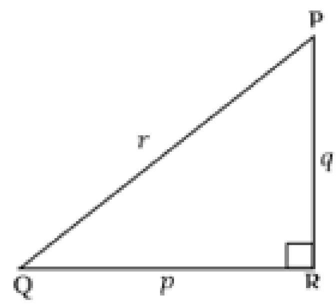
(b) $f(x)$ is discontinuous at $x = -2, x = 0$ and $x = 2$ hence not differentiable. **[1]**

Question 6 **[2]**

(i) Given, In ΔPQR, ∠R = 90°

$\therefore r^2 = p^2 + q^2$ ….. (i) (Pythagoras Theorem)

Now,



$tan^{-1}\left(\frac{\frac{p}{q+r} + \frac{q}{r+p}}{1 - \frac{pq}{(q+r)(r+p)}}\right)$

$= tan^{-1}\left(\frac{pr + p^2 + q^2 + qr}{qr + pq + r^2 + pr - pq}\right)$

$= tan^{-1}\left(\frac{pr + r^2 + qr}{pr + r^2 + qr}\right)$ ….. (from (i))

$= tan^{-1}(1)$

$= \frac{\pi}{4}$

OR

(ii) Given, $ax^2 + sin^{-1}(x^2 - 2x + 2) + cos^{-1}(x^2 - 2x + 2) = 0$

$\therefore ax^2 + \frac{\pi}{2} = 0$ ………………… (i) $\left[\because sin^{-1}x + cos^{-1}x = \frac{\pi}{2}\right]$

For the given inverse functions: $-1 \le x^2 - 2x + 2 \le 1$

Now, $(x^2 - 2x + 2) = (x-1)^2 + 1$

Since $(x-1)^2 \ge 0$

Minimum value occurs when $x = 1$

$\therefore (x-1)^2 + 1 \ge 1$

To satisfy domain ≤ 1 we have,

$(x-1)^2 + 1 = 1$

$\therefore (x-1)^2 = 0$

$\therefore x = 1$

Thus, only one value of x exists

Hence, roots are equal at $x = 1$

$\therefore a(1)^2 + \frac{\pi}{2} = 0$

$\therefore a = -\frac{\pi}{2}$

Question 7 **[2]**

(i) Vector projection of B on A

$$= (\vec{B}.\hat{A})\hat{A}$$
$$= \left(\frac{\vec{B}.\vec{A}}{|\vec{A}|}\right)\hat{A}$$
$$= \left(\frac{\vec{B}.\vec{A}}{|\vec{A}|^2}\right)\vec{A}$$
$$= -\frac{4}{9}(\hat{\imath} - 2\hat{\jmath} - 2\hat{k})$$

Scalar component of $\vec{B}$ on $\vec{A}$

$$= \vec{B}.\frac{\vec{A}}{|\vec{A}|}$$
$$= (6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}).\left(\frac{\hat{\imath}}{3} - \frac{2\hat{\jmath}}{3} - \frac{2}{3}\hat{k}\right)$$
$$= -\frac{4}{3}$$

OR

(ii) A ($\vec{a}$) = $\hat{\imath} + \hat{\jmath} + \hat{k}$, B ($\vec{b}$) = $2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$, B ($\vec{c}$) = $7\hat{\imath} + 4\hat{\jmath} + 9\hat{k}$

$\therefore \overrightarrow{AB} = \hat{\imath} + 2\hat{\jmath} - 5\hat{k}$ and $\overrightarrow{AC} = 6\hat{\imath} + 3\hat{\jmath} + 8\hat{k}$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -5 \\ 6 & 3 & 8 \end{vmatrix}$$
$$= 31\hat{\imath} - 38\hat{\jmath} - 9\hat{k}$$

$$|31\hat{\imath} - 38\hat{\jmath} - 9\hat{k}| = \sqrt{2486}$$

Unit vector = $\frac{31\hat{\imath} - 38\hat{\jmath} - 9\hat{k}}{\sqrt{2486}}$

Question 8

(i) Position vector of IMA(I) relative to Clock Tower $(\overrightarrow{OI}) = 3\hat{\imath} + 9\hat{\jmath}$ **[1]**

Position vector of Rajpur (R) relative to Clock Tower $(\overrightarrow{OR}) = 8\hat{\imath} + 4\hat{\jmath}$

Position vector of Bus stop(S) relative to Clock Tower $(\overrightarrow{OS}) = \frac{2}{3}(3\hat{\imath} + 9\hat{\jmath})$

$$= 2\hat{\imath} + 6\hat{\jmath}$$

(ii) Now, $\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = 2\hat{\imath} + 6\hat{\jmath} - 8\hat{\imath} - 4\hat{\jmath} = -6\hat{\imath} + 2\hat{\jmath}$. **[1]**

$\therefore \overrightarrow{OS}.\overrightarrow{RS} = -12 + 12 = 0.$

$\Rightarrow$ Bus stop is the closest point to Rajpur on the Clock Tower to IMA Road.

SECTION C – 21 MARKS

Question 9

(i) $2x + y \leq 9$ **[1]**

$y \geq x$

$x \geq 1$

(ii) $Z = 8 \; at \; C \; (1,7)$ **[1]**

(iii) $2x + y \geq 9$ **[1]**

Question 10 **[3]**

$\frac{dx}{dz} = 3$, and $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \rightarrow \frac{dy}{dz} = 3\frac{dy}{dx}$

$\frac{d^2y}{dz^2} = 3\frac{d}{dz}\left(\frac{dy}{dx}\right)$

$= 3\frac{d}{dx}\left(\frac{dy}{dx}\right).\frac{dx}{dz}$

$= 3\frac{d^2y}{dx^2}.3$

$\frac{d^2y}{dz^2} = 9\frac{d^2y}{dx^2}$

Question 11

(i) Simplify the given function: **[2]**

$f(x) = \log\left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}.\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}\right] + \tan^{-1}\frac{2x}{1-x^2}$

$f(x) = \log\left[\frac{\left(\sqrt{1+x^2}+x\right)^2}{1+x^2-x^2}\right] + 2\tan^{-1}x$

$f(x) = 2\log\left(\sqrt{1+x^2}+x\right) + 2\tan^{-1}x$

$f'(x) = 2\frac{1}{\sqrt{1+x^2}+x}.\left(\frac{2x}{2\sqrt{1+x^2}}+1\right) + \frac{2}{1+x^2}$

$f'(x) = \frac{2}{\sqrt{1+x^2}} + \frac{2}{1+x^2}$

(ii) The claim is incorrect. **[1]**

Since $x^2 \geq 0$, both $\frac{2}{\sqrt{1+x^2}}$ and $\frac{2}{1+x^2}$ are always positive.

Thus, $f'(x) > 0$ for all real x.

A local maximum requires $f'(x)$ to be zero and change sign. Here, the function is strictly increasing.

Question 12

(i) Let γ be the angle made by $\vec{n}$ with z-axis, then direction cosines of $\vec{n}$ are **[3]**

$l = cos\,45° = \frac{1}{\sqrt{2}}, m = cos\,60° = \frac{1}{2}$ and n = cosγ

$\because l^2 + m^2 + n^2 = 1$

$n^2 = \frac{1}{4}$

$n = \frac{1}{2}$ (neglecting $n = -\frac{1}{2}$ as γ is acute)

$\because |\vec{n}| = 8$

$\therefore \vec{n} = |\vec{n}|\left(l\hat{\imath} + m\hat{\jmath} + n\hat{k}\right)$

$\vec{n} = 8\left(\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{2} + \frac{\hat{k}}{2}\right)$

$= 4\sqrt{2}\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$

$\therefore$ Required plane passes through the point $\left(\sqrt{2}, -1, 1\right)$ having position vector

$\vec{a} = \sqrt{2}\hat{\imath} - \hat{\jmath} + \hat{k}$

$\therefore$ Vector equation is $(\vec{r} - \vec{a}).\vec{n} = 0$

$\vec{r}.\left(4\sqrt{2}\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\right) = \left(\sqrt{2}\hat{\imath} - \hat{\jmath} + \hat{k}\right).\left(4\sqrt{2}\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\right)$

$\Rightarrow \vec{r}.\left(4\sqrt{2}\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\right) = 8$

$\Rightarrow \vec{r}.\left(\sqrt{2}\hat{\imath} + \hat{\jmath} + \hat{k}\right) = 2$

OR

(ii) (a) $\pi_1: 2x + 6y - 2z = 5$ **[1]**

Substituting the point P $\left(2, \frac{1}{2}, 1\right)$,

LHS = $2 \times 2 + 6 \times \frac{1}{2} - 2 \times 1 = 4 + 3 - 2 = 5 = RHS$

(b) $\pi_2: 3x + 9y + pz = -\dfrac{51}{2}$ **[1]**

Given that $\pi_2 \parallel \pi_1 \Rightarrow \frac{3}{2} = \frac{9}{6} = \frac{p}{-2}$

$\therefore p = -3$

(c) Equation of the line that is perpendicular to π_1 is **[1]**

$\frac{x-2}{1} = \frac{y-\frac{1}{2}}{3} = \frac{z-1}{-1} = \lambda$

General points on the line are $(\lambda + 2, 3\lambda + \frac{1}{2}, -\lambda + 1)$

$\because Q \text{ lies on } \pi_2$

$\Rightarrow 3(\lambda + 2) + 9\left(3\lambda + \frac{1}{2}\right) - 3(1 - \lambda) = -\frac{51}{2}$

$\Rightarrow \lambda + 2 + 9\lambda + \frac{3}{2} - 1 + \lambda = -\frac{17}{2}$

$\Rightarrow 11\lambda = -11 \Rightarrow \lambda = -1$

$\therefore$ Co-ordinates of Q $(1, -\frac{5}{2}, 2)$

Question 13 **[3]**

(i)
$$\Delta = \frac{1}{xyz}\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$ Applied the property $|A| = |A^t|$

$$= -\begin{vmatrix} y^2 & x^2 & z^2 \\ y^3 & x^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} y^2 & z^2 & x^2 \\ y^3 & z^3 & x^3 \\ 1 & 1 & 1 \end{vmatrix}$$ (Hence proved)

$$\Delta=\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Delta=\begin{vmatrix} x^2-y^2 & x^3-y^3 & 0 \\ y^2-z^2 & y^3-z^3 & 0 \\ z^2 & z^3 & 1 \end{vmatrix} = (x-y)(y-z)\begin{vmatrix} x+y & x^2+y^2+xy & 0 \\ y+z & y^2+z^2+yz & 0 \\ z^2 & z^3 & 1 \end{vmatrix}$$

Again $R_2 \to R_2 - R_1$

$$= (x-y)(y-z)\begin{vmatrix} x+y & x^2+y^2+xy & 0 \\ z-x & z^2-x^2+y(z-x) & 0 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)\begin{vmatrix} x+y & x^2+y^2+xy & 0 \\ 1 & x+y+z & 0 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

$= (x-y)(y-z)(z-x)$ (As $xy+yz+zx=1$)

OR

(ii) Let Q be the quantities of chair ribbon and table cover needed

$\therefore Q = [150 \quad 20]$

Let P represents the prices of chair ribbon and table cover from each company.

$$\therefore P = \begin{array}{cc} & \\ \begin{bmatrix} 12 & 10 \\ 180 & 200 \end{bmatrix} & \begin{array}{l} \leftarrow \text{chair ribbons} \\ \leftarrow \text{table cover} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{Company A} & \text{Company B} \end{array} & \end{array}$$

$\therefore$ Total cost matrix is given by

$C = QP$

$\therefore C = [150 \quad 20]\begin{bmatrix} 12 & 10 \\ 180 & 200 \end{bmatrix}$

$\therefore C = [150 \times 12 + 20 \times 180 \qquad 150 \times 10 + 20 \times 200]$

$\therefore C = [1800 + 3600 \qquad 1500 + 4000]$

$\therefore C = [5400 \qquad 5500]$

Thus, the price of company A (₹ 5400) is ₹ 100 less than the price of the company B (₹5500).

Question 14

(i) Given, $\int_1^5 3f(x)dx = 12$ **[1]**

$\Rightarrow \int_1^5 f(x)dx = 4$

$\Rightarrow -\int_1^5 f(x)dx = -4$

$\Rightarrow \int_5^1 f(x)dx = -4. \qquad \left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$

Hence proved.

(ii) $\displaystyle\int_1^2 (x + f(x))\, dx + \int_2^5 (x + f(x))dx$ **[2]**

$=\int_1^5 (x + f(x))\, dx$

$= \int_1^5 x\, dx + \int_1^5 f(x)dx$

$=\left[\frac{x^2}{2}\right]_2^5 + 4$

$= \frac{25}{2} - \frac{1}{2} + 4$

$= 16$

Question 15

(i) Given, L is the tangent to the graph of f at the origin O. **[3]**

Slope of the tangent at O $= f'(0)$

$$= \frac{2a^2 - 0}{\sqrt{a^2 - 0}} = 2a$$

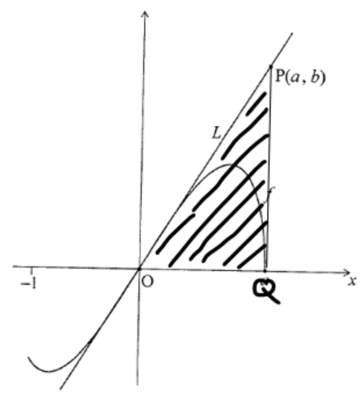
$\therefore$ Equation of L: $y - 0 = 2a(x - 0)$

$\Rightarrow y = 2ax.$

$\therefore \Delta OPQ = \int_0^a 2ax\, dx.$

$= 2a \times \left[\frac{x^2}{2}\right]_0^a$

$= a^3\ sq.units.$



OR

(ii) (a) Putting x=0 in $y = -3x + 9$, y=9 **[1]**

$\Rightarrow$P (0,9)

and Putting y=0 in $y = -3x + 9$, x=3

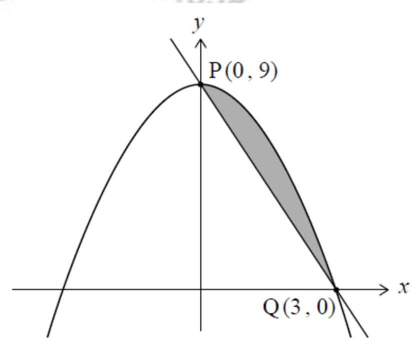
$\Rightarrow$ Q (3,0)

The parabola is of the form $y = ax^2 + c$, where a , $c \in \mathbb{Z}$

Satisfying by, (0,9), we get $c = 9$.

Satisfying by, (3,0), we get $a = -1$.

$\therefore$ Equation of parabola is: $y = -x^2 + 9$



(b) Equation of line PQ is: $\frac{x}{3} + \frac{y}{9} = 1$ i.e. $y = 9 - 3x$ **[2]**

$\therefore$ The area of the shaded region

$$= \int_0^3 (9 - x^2 - 9 + 3x)\, dx$$

$$= \int_0^3 (3x - x^2)\, dx$$

$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$

$= \frac{27}{2} - 9$

$= \frac{9}{2}$ sq units

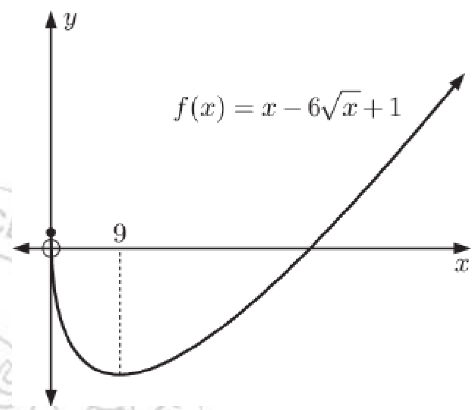
SECTION D – 25 MARKS

Question 16

(i) $f(x) = x - 6\sqrt{x} + 1$ **[1]**

Natural domain of $f(x)$ is $x \geq 0$.

i.e. $[0, \infty)$



(ii) As the graph fails the horizontal line test, hence it is not one-one. **[1]**

$\Rightarrow f(x)$ is not invertible.

$f(x) = x - 6\sqrt{x} + 1$

$\Rightarrow y = x - 6\sqrt{x} + 1$

$\Rightarrow \left(\sqrt{x}\right)^2 - 6\sqrt{x} + (1 - y) = 0$

$\therefore \sqrt{x} = \frac{6 \pm \sqrt{36 - 4(1-y)}}{2} = \frac{6 \pm 2\sqrt{y+8}}{2} = 3 \pm \sqrt{y+8}$

$\therefore x = \left(3 \pm \sqrt{y+8}\right)^2$

(iii) Given, $g(x) = x - 6\sqrt{x} + 1, 0 \leq x \leq 9$. **[1]**

$\therefore g^{-1}(x) = \left(3 - \sqrt{x+8}\right)^2$

$\therefore (g \circ g^{-1})(x) = \left(3 - \sqrt{x+8}\right)^2 - 6\left(3 - \sqrt{x+8}\right) + 1$

$= 9 - 6\sqrt{x+8} + x + 8 - 18 + 6\sqrt{x+8} + 1 = x$

$\therefore (g \circ g^{-1})(4) = 4$

(iv) Given, $h(x) = x - 6\sqrt{x} + 1, x \geq 9$. **[1]**

$\therefore h^{-1}(x) = \left(3 + \sqrt{x+8}\right)^2$

$\therefore (h \circ h^{-1})(x) = \left(3 + \sqrt{x+8}\right)^2 - 6\left(3 + \sqrt{x+8}\right) + 1$

$= 9 + 6\sqrt{x+8} + x + 8 - 18 - 6\sqrt{x+8} + 1 = x$

$\therefore (h \circ h^{-1})(16) = 16$

(v) When $g^{-1}(x) = h^{-1}(x)$ **[1]**

$\Rightarrow \left(3 - \sqrt{x+8}\right)^2 = \left(3 + \sqrt{x+8}\right)^2$

$\Rightarrow 9 - 6\sqrt{x+8} + x + 8 = 9 + 6\sqrt{x+8} + x + 8$

$\Rightarrow \sqrt{x+8} = 0$

$\Rightarrow x = -8$

Question 17

(i) (a) **[1]**

$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$

$f'(x) = x^2 - 4x + 3$

$f'(x) = 0 \Longrightarrow x^2 - 4x + 3 = 0 \Longrightarrow (x-1)(x-3) = 0 \Longrightarrow x = 1 \;\&\; x = 3$

Climbing up (increasing) $\Longrightarrow f'(x) > 0$ on $[0,1) \cup (3,\ \infty)$

Descending (decreasing) $\Longrightarrow f'(x) < 0$ on $(1, 3)$

(b) Slope of the horizontal tangent $f'(x) = 0$, at $x = 1 \;\&\; x = 3$ **[1]**

At $x = 1, f(1) = \frac{1}{3} - 2 + 3 + 5 = 6 \cdot 33$

At $x = 3, f(3) = 9 - 18 + 9 + 5 = 5$

Coordinates of the points on the curve: $(1, 6 \cdot 33)$ & $(3,\ 5)$

(c) Maxima & Minima in the interval $[0, 4]$ **[2]**

Evaluate the function at the critical points $x = 1$ *and* $x = 3$ and end points of the interval $x = 0$ *and* $x = 4$

$f(0) = 5$

$f(1) = 6 \cdot 33$

$f(3) = 5$

$f(4) = \frac{64}{3} - 32 + 12 + 5 = 6.33$

The absolute maximum is 6·33 at $x = 1$ *and* $x = 4$

The absolute minimum is 5 at $x = 0$ *and* $x = 3$

(d) The equation of normal at $x = 1$ **[1]**

Slope of the tangent at $x = 1 \Longrightarrow f'(1) = 0$

i.e. the tangent is a horizontal line $\Longrightarrow$ normal is a vertical line $(x = 1)$

OR

(ii) Let the radius, height, volume and semi vertical angle of the cone be $r, h, V \;\&\; \alpha$

$\tan\alpha = \frac{r}{h} = 0.5 \Longrightarrow r = 0.5\,h$

Rate of flow (drained out) $= \frac{dV}{dt} = -5m^3/min$

Volume $= V = \frac{1}{3}\pi r^2 h$

(a) $V = \frac{1}{3}\pi(0.5h)^2 h = \frac{0.25}{3}\pi h^3 = \frac{\pi}{12}h^3$ **[2]**

$\frac{dV}{dt} = \frac{\pi}{12}\ 3h^2\frac{dh}{dt} \Longrightarrow \frac{\pi}{4}h^2\frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{\pi}{4}h^2} = \frac{-5}{\frac{\pi}{4}h^2}$

Now substitute $\frac{dV}{dt} = -5 \; and \; h = 4$

$-5 = \frac{\pi}{4} 4^2 \frac{dh}{dt} = 4\pi \frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{-5}{4\pi} m/min$

(b) $l =$ Slant height $= \sqrt{r^2 + h^2} = \sqrt{(0.5h)^2 + h^2} = \sqrt{1.25h^2} = \frac{\sqrt{5}}{2} h$ **[1]**

Curved Surface area $= \pi r l$

$S = \pi (0.5h) \frac{\sqrt{5}}{2} h = \frac{\sqrt{5}}{4} \pi h^2$

$\frac{dS}{dh} = \frac{\sqrt{5}}{2} \pi h$

Since $h > 0$ for physical tank $\frac{dS}{dh} > 0$

$\Longrightarrow$ Surface area S is strictly increasing on$(0, \infty)$

(c) Given that Cost of the additive $C = kS^2 = k\left(\frac{\sqrt{5}}{4} \pi h^2\right)^2 = \frac{5k\pi^2}{16} h^4$ **[2]**

Rate of change of cost $= \frac{dC}{dt} = \frac{dC}{dh} . \frac{dh}{dt}$

$$= \frac{5k\pi^2}{4} h^3 . \frac{5}{\frac{\pi}{4} h^2} = 25k\pi h$$

The rate of cost change $\frac{dC}{dt}$ is a linear function of h.

The rate of change increases as h increases. Therefore, the cost is increasing most rapidly at the maximum possible depth h allowed by the tank.

Question 18

(i) (a) $\int \ln x \; dx$ **[2]**

$= \int \ln x . 1 \; dx$

I II

Applying by parts:

$= \ln x . \int 1 \, dx - \int \left\{ \int \frac{d}{dx} (\ln x) \int 1 dx \right\} dx$

$= \ln x \times x - \int \frac{1}{x} \times x dx$

$= x \ln x - \int 1 dx$

$= x(\ln x - 1) + c$

(b) $I = \int \frac{\ln\left[\ln\left(\frac{1+x}{1-x}\right)\right]}{1 - x^2} dx$ **[3]**

Let $\ln\left(\frac{1+x}{1-x}\right) = t$

Differentiating w.r.t't':

$\Rightarrow \frac{1-x}{1+x} \times \frac{1-x-(1+x)(-1)}{(1-x)^2} \times \frac{dx}{dt} = 1$

$\Rightarrow 2 \times \frac{1}{1-x^2} dx = dt$

$\therefore I = \int \frac{\ln\left[\ln\left(\frac{1+x}{1-x}\right)\right]}{1 - x^2} dx$

$= \frac{1}{2} \int \ln\left[\ln\left(\frac{1+x}{1-x}\right)\right] \times \frac{2}{1-x^2} dx$

$= \frac{1}{2}\int \ln t \,.dt$
$= \frac{1}{2}\ t(\ln t - 1) + c$
$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\left[\ln\left[\ln\left(\frac{1+x}{1-x}\right)\right] - 1\right] + c$

OR

(ii) (a) $\int \tan x\ dx$ **[2]**

$= \int \frac{\sin x}{\cos x}\ dx$
$= -\int \frac{d(\cos x)}{\cos x}$
$= -\ln|\cos x| + c$

(b) $I = \int \frac{dx}{\cot\frac{x}{2}\cot\frac{x}{3}\cot\frac{x}{6}}$ **[3]**

$= \int \tan\frac{x}{2}\tan\frac{x}{3}\tan\frac{x}{6}\,dx$
Now, $\frac{x}{2} = \frac{x}{3} + \frac{x}{6}$
$\Rightarrow \tan\frac{x}{2} = \tan\left(\frac{x}{3} + \frac{x}{6}\right)$
$\Rightarrow \tan\frac{x}{2} = \frac{\tan\frac{x}{3} + \tan\frac{x}{6}}{1 - \tan\frac{x}{3}\tan\frac{x}{6}}$
$\Rightarrow \tan\frac{x}{2}\tan\frac{x}{3}\tan\frac{x}{6} = \tan\frac{x}{2} - \tan\frac{x}{3} - \tan\frac{x}{6}$
$\therefore I = \int \tan\frac{x}{2}dx - \int \tan\frac{x}{3}dx - \int \tan\frac{x}{6}dx$
$= -2\ln\left|\cos\frac{x}{2}\right| + 3\ln\left|\cos\frac{x}{3}\right| + 6\ln\left|\cos\frac{x}{6}\right| + c$

Question 19

(i) P (laptop behind door A) = P (laptop behind door B) = P (laptop behind door C) = $\frac{1}{3}$

If the laptop is behind door A, either door B or door C can be chosen to be opened.
$\Longrightarrow$ P (Door B opened/ laptop behind A) = $\frac{1}{2}$
Since the host will never reveal the laptop
$\Longrightarrow$ P (Door B opened/ laptop behind B) = 0
If the laptop is behind door C, then only door B can be opened since door A is the selection of the contestant and cannot be opened.
$\Longrightarrow$P (Door B opened/ laptop behind C) = 1

(a) P (laptop is behind Door A/Door B has been opened) = $\frac{\frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3} + 0\times\frac{1}{3} + 1\times\frac{1}{3}} = \frac{1}{3}$ **[2]**

(b) P (laptop is behind Door C/Door B has been opened) = $\frac{1\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3}+0\times\frac{1}{3}+1\times\frac{1}{3}} = \frac{2}{3}$ **[2]**

(c) The contestant should change their selection to Door C as the probability is more. **[1]**

OR

(ii) A: Email is harmful

B: The filter marks email dangerous

Probability that the email is harmful = P(A) $= \frac{1}{1500}$

(a) Probability that the email is safe $= P(\bar{A}) = \frac{1499}{1500}$ **[1]**

(b) Probability that given the email is harmful the filter marks it dangerous $= P\left(\frac{B}{A}\right) = 0 \cdot 97$ **[2]**

Probability that the given email is safe and the filter marks it safe $= P\left(\frac{\bar{B}}{\bar{A}}\right) =$ 0.97

Probability that the given email is harmful and the filter marks it safe $= P\left(\frac{B}{\bar{A}}\right) = 1 - P\left(\frac{\bar{B}}{\bar{A}}\right) = 0 \cdot 03$

(c) Probability that filter marks the email dangerous (Total probability) **[2]**

$= P(B) = P\left(\frac{B}{A}\right)P(A) + P\left(\frac{B}{\bar{A}}\right)P(\bar{A}) = 0 \cdot 97 \times \frac{1}{1500} + 0 \cdot 03 \times \frac{1499}{1500}$

$= 0 \cdot 0306$

Probability that given the email is marked dangerous but it is safe =

$P\left(\frac{\bar{A}}{B}\right) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P\left(\frac{B}{\bar{A}}\right)P(\bar{A})}{P(B)} = \frac{P\left(\frac{B}{\bar{A}}\right)P(\bar{A})}{P\left(\frac{B}{A}\right)P(A)+P\left(\frac{B}{\bar{A}}\right)P(\bar{A})}$

$$= \frac{0 \cdot 03 \times \frac{1499}{1500}}{0 \cdot 97 \times \frac{1}{1500} + 0 \cdot 03 \times \frac{1499}{1500}} = 0 \cdot 9789 \approx 0 \cdot 98$$

Question 20 **[5]**

Given, $S(t) = at^2 + bt + c$

$$\therefore S(1) = a(1)^2 + b(1) + c \Rightarrow a + b + c = 285 \rightarrow (i)$$
$$S(2) = a(2)^2 + b(2) + c \Rightarrow 4a + 2b + c = 260 \rightarrow (ii)$$
$$S(4) = a(4)^2 + b(4) + c \Rightarrow 16a + 4b + c = 180 \rightarrow (iii)$$

The system can be written as $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 285 \\ 260 \\ 180 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and B $= \begin{bmatrix} 285 \\ 260 \\ 180 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = 1(2-4) - 1(4-16) + 1(16-32)$$

$$= -2 + 12 - 16 = -6 \neq 0$$

$\therefore A^{-1}$ exists and $A^{-1} = \frac{1}{|A|}$. adj.A.

$$\text{adj.A} = \begin{bmatrix} -2 & 3 & -1 \\ 12 & -15 & 3 \\ -16 & 12 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj.A} = -\frac{1}{6} \begin{bmatrix} -2 & 3 & -1 \\ 12 & -15 & 3 \\ -16 & 12 & -2 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -2 & 3 & -1 \\ 12 & -15 & 3 \\ -16 & 12 & -2 \end{bmatrix}\begin{bmatrix} 285 \\ 260 \\ 180 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -570 + 780 - 180 \\ 3420 - 3900 + 540 \\ -4560 + 3120 - 360 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} 30 \\ 60 \\ -1800 \end{bmatrix}$$

$\therefore a = -5, b = -10$ and $c = 300$

$\therefore$ Height equation: $S(t) = -5t^2 - 10t + 300$

Height of the cliff:

At $t = 0$, $\quad S(0) = -5(0)^2 - 10(0) + 300$

$\therefore$ Height of the cliff $= 300$ m.